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Application of Γ-fuzzy soft relation in decision-making

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Abstract. This study explores the extension of $I = [0,1]$-fuzzy soft relations to Γ-fuzzy soft relations using the $\Gamma = [0,1]^{|E|}$-lattice. When the parameter set E contains only one parameter, $\Gamma = I = [0,1]$, resulting in applications similar to those found in conventional fuzzy scenarios. However, when E contains at least two parameters, the applications resemble typical fuzzy soft situations where $I = [0,1]$ fuzzifies the soft set. It is posited that the concept of an "$I = [0,1]$"-fuzzy soft relation naturally extends to a "$\Gamma = [0,1]^{|E|}$"-fuzzy soft relation. This extension is significant because it considers the individual impact of each parameter element in the subset A of the parameter set E, rather than simultaneously assessing the effects of all parameters. This generalization is illustrated, and its relevance to decision-making-related problems is demonstrated through an application example.

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1. Introduction

In his seminal work [1], Zadeh introduced the concept of a fuzzy set as a generalization of classical crisp sets, which has since found applications in a wide variety of fields involving uncertainty and imprecision. To address a different aspect of uncertainty, Molodtsov proposed the notion of a soft set in [2], which became a powerful framework for modeling problems where parametrization plays a central role. The foundational ideas of soft sets have since been applied across mathematics, computer science, economics, medical sciences, and decision-making.

Building upon these foundational concepts, Maji et al. combined fuzzy sets and soft sets to develop the notion of *fuzzy soft sets* [3], which added a nuanced layer of membership-based reasoning to parameterized environments. This was further extended by Majumdar and Samanta in [4], where fuzzy soft sets were explored in the context of decision-making problems. A comparison of soft sets, fuzzy sets, and rough sets was conducted by Aktaş and Çağman in [5], highlighting their respective strengths and intersections. Additionally, Yang et al. in [6] contributed to the algebraic structure of fuzzy soft sets by defining and exploring various operations.

A fuzzy relation is classically defined as a fuzzy subset of the Cartesian product of crisp sets. When two such sets are involved, the result is a binary fuzzy relation, see [7] for foundational concepts.

Decision-making remains a central and pressing task across all fields. As such, fuzzy sets, soft sets, and their combinations (such as fuzzy soft sets) have become instrumental in constructing decision-making models, especially where vague or incomplete information is involved. For practical applications in decision-making and related topics, see [8, 9]. In [10], Dusmanta Kumar Sut introduced the application of fuzzy soft relation in Decision-making using the membership values to compute maximum score value, which determine decision, but in [11], Roy and Maji introduced the application of fuzzy soft set in Decision-making, and used the same procedure, which we are using in this paper. For broader discussions on soft sets and applications, refer to [12, 13, 14, 15, 16]; and for background on lattice theory, see [17].

In recent years, fuzzy soft sets and relations have been extended and applied in diverse decision-making environments. [18, 19, 20, 21, 22, 23] are some representative modern contributions in this direction:

In this study, we introduce the concept of a Γ-lattice, formed by combining lattices indexed by parameters in a set E. This framework allows us to represent not only the global structure induced by the parameter set E but also the localized influence of each individual parameter. As an extension of the classical fuzzy soft relation (defined over $I = [0, 1]$), we propose the Γ-fuzzy soft relation, where $\Gamma = [0, 1]^{|E|}$ allows vector-valued membership degrees indexed by parameters.

This extension addresses a key limitation of classical models: the inability to preserve parameter-specific contributions in multi-criteria settings. By capturing this additional structure, the proposed framework enhances parametric specialization and interpretability. Given the critical role of structured, transparent reasoning in decision-making, the Γ-fuzzy soft relation is particularly well-suited for decision-support systems.

2. Γ-fuzzy soft relation

Definition 2.1 ([2])**.** The pair (F, A), where F maps a subset A of E to the power set $P(U)$ of an initial universe U, is termed as a soft set over U.

Definition 2.2 ([3])**.** A pair (G, A) is termed a fuzzy soft set, if $G : A \to I^U$, where A is a subset of E, and G represents a mapping from A to the family I^U, encompassing all fuzzy subsets of U, where I represents the closed unit interval $[0, 1]$.

Definition 2.3 ([10, 16])**.** Let X and Y represent two initial universal sets and E be the set of parameters, (F, A) and (G, A) denote two fuzzy soft sets over X and

Y, respectively. Let $I^{X\times Y}$ denote the set of all fuzzy subsets of $X\times Y$. Then (H, A) constitutes a *fuzzy soft relation between* (F, A) *and* (G, A) *over* $X \times Y$, if H is a mapping $H : A \to I^{X\times Y}$, defined as follows: for all $e \in A$ and all $(x, y) \in X \times Y$,

$$H(e)(x, y) = F(e)(x) \wedge G(e)(y).$$

Definition 2.4. If $\Gamma = \prod_{e\in E} I_e$ with $I_e = I = [0, 1]$ for all $e \in E$, then the Γ-fuzzy soft relation represents a generalization of the I-fuzzy soft relation.

Example 2.5. Let $E = \{e_1, e_2\}$ be a set of parameters, and $X = \{x_1, x_2\}, Y = \{y_1, y_2\}$ be two sets. Let $I = [0, 1]$, so we define $\Gamma = I \times I = [0, 1]^2$, which means each membership value in the relation is a vector in Γ indexed by (e_1, e_2).

(1) **The classical I-fuzzy soft relation.**

Let the I-fuzzy soft sets (F, E) and (G, E) be:

$$\begin{array}{llll} F(e_1)(x_1) = 0.4, & F(e_1)(x_2) = 0.8 & G(e_1)(y_1) = 0.7, & G(e_1)(y_2) = 0.6 \\ F(e_2)(x_1) = 0.5, & F(e_2)(x_2) = 0.7 & G(e_2)(y_1) = 0.6, & G(e_2)(y_2) = 0.5 \end{array}$$

Then the I-fuzzy soft relation $H : E \to I^{X\times Y}$ from (F, E) to (G, E) is defined by:

$$H(e)(x_i, y_j) = \min(F(e)(x_i), G(e)(y_j))$$

$H(e_1)$	y_1	y_2
x_1	$\min(0.4, 0.7) = 0.4$	$\min(0.4, 0.6) = 0.4$
x_2	$\min(0.8, 0.7) = 0.7$	$\min(0.8, 0.6) = 0.6$

$H(e_2)$	y_1	y_2
x_1	$\min(0.5, 0.6) = 0.5$	$\min(0.5, 0.5) = 0.5$
x_2	$\min(0.7, 0.6) = 0.6$	$\min(0.7, 0.5) = 0.5$

(2) Let the Γ-fuzzy soft sets (F, E) and (G, E) be defined by:

$$\begin{array}{ll} F(e_1)(x_1) = (0.4, 0.6), & G(e_1)(y_1) = (0.7, 0.5), \\ F(e_1)(x_2) = (0.8, 0.3), & G(e_1)(y_2) = (0.6, 0.2), \end{array}$$

$$\begin{array}{ll} F(e_2)(x_1) = (0.5, 0.7), & G(e_2)(y_1) = (0.6, 0.6), \\ F(e_2)(x_2) = (0.7, 0.4), & G(e_2)(y_2) = (0.5, 0.3). \end{array}$$

Then the Γ-fuzzy soft relation $H : E \to (I \times I = \Gamma)^{X\times Y}$ from (F, E) to (G, E) is defined by:

$$H(e)(x_i, y_j) = \min(F(e)(x_i), G(e)(y_j)) \quad \text{(component-wise minimum)}$$

$H(e_1)$	y_1	y_2
x_1	$\min((0.4, 0.6), (0.7, 0.5)) = (0.4, 0.5)$	$\min((0.4, 0.6), (0.6, 0.2)) = (0.4, 0.2)$
x_2	$\min((0.8, 0.3), (0.7, 0.5)) = (0.7, 0.3)$	$\min((0.8, 0.3), (0.6, 0.2)) = (0.6, 0.2)$

$H(e_2)$	y_1	y_2
x_1	$\min((0.5, 0.7), (0.6, 0.6)) = (0.5, 0.6)$	$\min((0.5, 0.7), (0.5, 0.3)) = (0.5, 0.3)$
x_2	$\min((0.7, 0.4), (0.6, 0.6)) = (0.6, 0.4)$	$\min((0.7, 0.4), (0.5, 0.3)) = (0.5, 0.3)$

This Example illustrate that each Γ-fuzzy soft relation value $H(e)(x_i, y_j)$ is now a vector in $\Gamma = [0,1]^2$, where each component reflects the parameter-specific membership. This illustrates how the Γ-fuzzy soft relation retains full parametric structure and generalizes the classical I-fuzzy soft relation which would only assign a single scalar value.

It's evident that for any parameter $e_i \in E$, both $F(e_i) : X \to \Gamma$ and $G(e_i) : Y \to \Gamma$ represent Γ-fuzzy subsets of X and Y respectively. Furthermore, all mappings $F(e_i)(x), G(e_i)(y), H(e_i)((x,y) : E \to \bigcup_{e\in E} I_e = I$ are I-fuzzy subsets of E.

Definition 2.6. For any $A \subset E$, consider two Γ-fuzzy soft sets (F, A) and (G, A) over X and Y respectively, along with the Γ-fuzzy soft relation (H, A) between them. For any $e_i \in A$, the ordered pairs $(F(e_i), X)$ and $(G(e_i), Y)$ are termed the i-projections of the fuzzy soft sets (F, A) and (G, A) respectively. Additionally, $(H(e_i), X \times Y)$ is referred to as the i-projection of the fuzzy soft relation (H, A).

Let $X = \{x_1, x_2, x_3, \ldots, x_n\}$, $Y = \{y_1, y_2, y_3, \ldots, y_m\}$, and $A \subset E$, where $A = \{e_1, e_2, e_3, \ldots, e_r\}$. Then, the tabular forms for representing i-projections of fuzzy soft sets, i-projections of fuzzy soft relations, and comparison tables for i-projections of fuzzy soft relations, for all $e_i \in A$, are constructed as shown in Table 1, Table 2, Table 3, and Table 4.

Table 1. i-Projections $(F(e_i), X),\ 1 \le i \le r$.

X/E	e_1	e_2	$\ldots$	e_r
x_1	$F(e_i)(x_1)(e_1)$	$F(e_i)(x_1)(e_2)$	$\ldots$	$F(e_i)(x_1)(e_r)$
x_2	$F(e_i)(x_2)(e_1)$	$F(e_i)(x_2)(e_2)$	$\ldots$	$F(e_i)(x_2)(e_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_n	$F(e_i)(x_n)(e_1)$	$F(e_i)(x_n)(e_2)$	$\ldots$	$F(e_i)(x_n)(e_r)$

Table 2. i-Projections $(G(e_i), Y), 1 \le i \le r$.

Y/E	e_1	e_2	$\ldots$	e_r
y_1	$G(e_i)(y_1)(e_1)$	$G(e_i)(y_1)(e_2)$	$\ldots$	$G(e_i)(y_1)(e_r)$
y_2	$G(e_i)(y_2)(e_1)$	$G(e_i)(y_2)(e_2)$	$\ldots$	$G(e_i)(y_2)(e_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
y_m	$G(e_i)(y_m)(e_1)$	$G(e_i)(y_m)(e_2)$	$\ldots$	$G(e_i)(y_m)(e_r)$

3. Application of Γ-fuzzy soft relation in decision making

Consider two sets: $X = \{x_1, x_2, x_3, \ldots, x_n\}$ and $Y = \{y_1, y_2, y_3, \ldots, y_m\}$, along with the set of parameters E. The problem at hand involves selecting a mixed pair of two elements, one from each set X and Y, based on parameters in the set $A = \{e_1, e_2, e_3, \ldots, e_r\} \subset E$. Initially, there are mn mixed pairs to choose from for each parameter in the set A. However, the following algorithm simplifies the

TABLE 3. i-Projections $(H(e_i), X \times Y), 1 \le i \le r$.

$(X \times Y)/E$	e_1	e_2	$\dots$	e_r
(x_1, y_1)	$H(e_i)(x_1, y_1)(e_1)$	$H(e_i)(x_1, y_1)(e_2)$	$\dots$	$H(e_i)(x_1, y_1)(e_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
(x_1, y_m)	$H(e_i)(x_1, y_m)(e_1)$	$H(e_i)(x_1, y_m)(e_2)$	$\dots$	$H(e_i)(x_1, y_m)(e_r)$
(x_2, y_1)	$H(e_i)(x_2, y_1)(e_1)$	$H(e_i)(x_2, y_1)(e_2)$	$\dots$	$H(e_i)(x_2, y_1)(e_r)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
(x_n, y_m)	$H(e_i)(x_n, y_m)(e_1)$	$H(e_i)(x_n, y_m)(e_2)$	$\dots$	$H(e_i)(x_n, y_m)(e_r)$

TABLE 4. Comparison table for $(H(e_i), X \times Y), 1 \le i \le r$.

$X \times Y$	(x_1, y_1)	(x_1, y_2)	$\dots$	(x_1, y_m)	(x_2, y_1)	$\dots$	(x_2, y_m)	$\dots$	(x_n, y_m)
(x_1, y_1)	$r^{x_1}_{11}$	$r^{x_1}_{12}$	$\dots$	$r^{x_1}_{1m}$	$r^{x_1}_{1(m+1)}$	$\dots$	$r^{x_1}_{1(2m)}$	$\dots$	$r^{x_1}_{1(mn)}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
(x_1, y_m)	$r^{x_1}_{m1}$	$r^{x_1}_{m2}$	$\dots$	r_{mm}	$r^{x_1}_{m(m+1)}$	$\dots$	$r^{x_1}_{m(2m)}$	$\dots$	$r^{x_1}_{m(mn)}$
(x_2, y_1)	$r^{x_2}_{11}$	$r^{x_2}_{12}$	$\dots$	$r^{x_2}_{1m}$	$r^{x_2}_{1(m+1)}$	$\dots$	$r^{x_2}_{1(2m)}$	$\dots$	$r^{x_2}_{1(mn)}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
(x_2, y_m)	$r^{x_2}_{(2m)1}$	$r^{x_2}_{(2m)2}$	$\dots$	$r^{x_2}_{(2m)m}$	$r^{x_2}_{(2m)(m+1)}$	$\dots$	$r^{x_2}_{(2m)(2m)}$	$\dots$	$r^{x_2}_{(2m)(mn)}$
(x_3, y_1)	$r^{x_3}_{11}$	$r^{x_3}_{12}$	$\dots$	$r^{x_3}_{1m}$	$r^{x_3}_{1(m+1)}$	$\dots$	$r^{x_3}_{1(2m)}$	$\dots$	$r^{x_3}_{1(mn)}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
(x_n, y_m)	$r^{x_n}_{(nm)1}$	$r^{x_n}_{(nm)2}$	$\dots$	$r^{x_n}_{(nm)m}$	$r^{x_n}_{(nm)(m+1)}$	$\dots$	$r^{x_n}_{(nm)(2m)}$	$\dots$	$r^{x_n}_{(nm)(mn)}$

selection process, reducing the choices to only r mixed pairs, corresponding to the parameters in the set A.

3.1. **Algorithm for Selection of Mixed Pairs.**

Input:
- Two Γ-fuzzy soft sets (F, A) and (G, A) representing male and female managerial candidates.
- Γ-fuzzy soft relation (H, A) between (F, A) and (G, A), which measures the compatibility of pairs based on various managerial skills.
- Parameters e_i where $e_i \in A$, representing different skills and attributes such as technical expertise, project management, and communication.

Output: Suitable mixed pairs based on the selection criteria for forming a strong management team.

Step 1: Define the Γ-fuzzy soft sets (F, A) and (G, A) according to the, given rules, where F represents the male candidates, and G represents the female candidates. The comparison value $r^{x_l}_{pj}$ between the pairs (x_h, y_d) (row) and (x_f, y_z) (column) is defined as:

Step 2: Construct the Γ-fuzzy soft relation (H, A) between (F, A) and (G, A), which represents the compatibility of each pair based on the parameters defined.

Step 3: For each $e_i \in A$, form the i-projections $(F(e_i), X)$ and $(G(e_i), Y)$, which represent the evaluations of the male and female candidates, respectively, on the parameter e_i.

Step 4: For each $e_i \in A$, create the i-projections $(H(e_i), X \times Y)$ of the Γ-fuzzy soft relation (H, A), which shows the compatibility of male and female candidates based on e_i.

Step 5: For each $e_i \in A$, construct a comparison table corresponding to the i-projection $(H(e_i), X \times Y)$. The comparison value $r_{pj}^{x_l}$ between the pairs (x_h, y_d) (in row) and (x_f, y_z) (in column), where $p, f, h \in \{1, 2, \ldots, n\}$, $j, d, z \in \{1, 2, \ldots, m\}$ and $A = \{e_1, e_2, \ldots, e_r\}$ is defined as: $r_{pj}^{x_l} = \sum_{k=1}^{r} \left(c_{pj}^{x_l}\right)^k$, where

$$\left(c_{pj}^{x_l}\right)^k = \begin{cases} 1, & \text{if } H(e_i)(x_h, y_d)(e_k) \geq H(e_i)(x_f, y_z)(e_k), \\ 0, & \text{otherwise.} \end{cases}$$

Step 6: For each $e_i \in A$, compute the row-sums $W^i = \sum r_{pj}^{x_l}$ and column-sums $C^i = \sum r_{jp}^{x_l}$ of the comparison tables.

Step 7: For each $e_i \in A$, compute the score values $S^i = W^i - C^i$, which represent the relative suitability of each mixed pair for the management team.

Step 8: Find the maximum value of S^i, denoted as $\max\{S^i\}$, for all $e_i \in A$.

Step 9: Select the good mixed pair based on the following criteria:
- If the selection is based solely on the parameter e_i, choose the maximum value of S^i.
- If the selection is based on all parameters $e_i \in A$, choose the best pair(s) from the r pairs instead of selecting from all possible mn mixed pairs.

Step 10: Output the selected good mixed pairs according to the chosen criteria, which can be used for forming a balanced managerial team with complementary skills.

3.2. Application Example: Selecting a Management Team for a Computer Company

Let $X = \{x_1, x_2, x_3, x_4\}, Y = \{y_1, y_2, y_3\}$ represent sets of male and female candidates for managerial positions in a computer company, respectively. Define the set of evaluation parameters $E = \{e_1, e_2, e_3, e_4, e_5\}$, where

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix} \equiv \begin{bmatrix} \texttt{Technical Expertise} \\ \texttt{Project Management Skills} \\ \texttt{Innovation and Creativity} \\ \texttt{Leadership and Team Management} \\ \texttt{Communication and Collaboration Skills} \end{bmatrix}.$$

Let $A = \{e_1, e_2, e_3\}$ be a subset of E, consisting of the most important parameters considered by the HR (Human Resources) department for forming an effective management team in the computer company.

For each parameter $e \in E$, let $I_e = I = [0, 1]$, meaning each criterion is rated on a scale from 0 to 1, where 0 indicates poor performance and 1 indicates excellent performance. The combined evaluation space is then given by $\Gamma = \prod_{e \in E} I_e = I^5$.

The HR department applies the Algorithm 3.1 to select a balanced team of managers:

Step 1: Define the Γ-fuzzy soft sets (F, A) and (G, A) According to the "CV" of managerial pairs, the HR creates two Γ-fuzzy soft sets (F, A) and (G, A) over X and Y, respectively as follows:

$$\begin{array}{ll}
F(e_1)(x_1) = (0.3, 0.6, 0.4, 0, 0), & G(e_1)(y_1) = (0.2, 0.6, 0.5, 0, 0), \\
F(e_1)(x_2) = (0.4, 0.5, 0.8, 0, 0), & G(e_1)(y_2) = (0.9, 0.5, 0.3, 0, 0), \\
F(e_1)(x_3) = (0.1, 0.5, 0.2, 0, 0), & G(e_1)(y_3) = (0.6, 0.5, 0.1, 0, 0), \\
F(e_1)(x_4) = (0.9, 0.7, 0.5, 0, 0), & \\
F(e_2)(x_1) = (0.1, 0.3, 0.5, 0, 0), & G(e_2)(y_1) = (0.8, 0.7, 0.9, 0, 0), \\
F(e_2)(x_2) = (0.8, 0.6, 0.4, 0, 0), & G(e_2)(y_2) = (0.4, 0.3, 0.6, 0, 0), \\
F(e_2)(x_3) = (0.3, 0.5, 0.6, 0, 0), & G(e_2)(y_3) = (0.7, 0.9, 0.4, 0, 0), \\
F(e_2)(x_4) = (0.5, 0.9, 0.7, 0, 0), & \\
F(e_3)(x_1) = (0.3, 0.5, 0.7, 0, 0), & G(e_3)(y_1) = (0.2, 0.5, 0.6, 0, 0), \\
F(e_3)(x_2) = (0.5, 0.7, 0.9, 0, 0), & G(e_3)(y_2) = (0.2, 0.9, 0.5, 0, 0), \\
F(e_3)(x_3) = (0.5, 0.9, 0.3, 0, 0), & G(e_3)(y_3) = (0.9, 0.5, 0.7, 0, 0), \\
F(e_3)(x_4) = (0.7, 0.1, 0.5, 0, 0). &
\end{array}$$

Step 2: Form the Γ-fuzzy soft relation (H, A) between (F, A) and (G, A):

$$\begin{array}{ll}
H(e_1)(x_1, y_1) = (0.2, 0.6, 0.4, 0, 0), & H(e_1)(x_1, y_2) = (0.3, 0.5, 0.3, 0, 0), \\
H(e_1)(x_1, y_3) = (0.3, 0.5, 0.1, 0, 0), & H(e_1)(x_2, y_1) = (0.2, 0.5, 0.5, 0, 0), \\
H(e_1)(x_2, y_2) = (0.4, 0.5, 0.3, 0, 0), & H(e_1)(x_2, y_3) = (0.4, 0.5, 0.1, 0, 0), \\
H(e_1)(x_3, y_1) = (0.1, 0.5, 0.2, 0, 0), & H(e_1)(x_3, y_2) = (0.1, 0.5, 0.2, 0, 0), \\
H(e_1)(x_3, y_3) = (0.1, 0.5, 0.1, 0, 0), & H(e_1)(x_4, y_1) = (0.2, 0.6, 0.5, 0, 0), \\
H(e_1)(x_4, y_2) = (0.9, 0.5, 0.3, 0, 0), & H(e_1)(x_4, y_3) = (0.6, 0.5, 0.1, 0, 0), \\
H(e_2)(x_1, y_1) = (0.1, 0.3, 0.5, 0, 0), & H(e_2)(x_1, y_2) = (0.1, 0.3, 0.5, 0, 0), \\
H(e_2)(x_1, y_3) = (0.1, 0.3, 0.4, 0, 0), & H(e_2)(x_2, y_1) = (0.8, 0.6, 0.4, 0, 0), \\
H(e_2)(x_2, y_2) = (0.4, 0.3, 0.4, 0, 0), & H(e_2)(x_2, y_3) = (0.7, 0.6, 0.4, 0, 0), \\
H(e_2)(x_3, y_1) = (0.3, 0.5, 0.6, 0, 0), & H(e_2)(x_3, y_2) = (0.3, 0.3, 0.6, 0, 0), \\
H(e_2)(x_3, y_3) = (0.3, 0.5, 0.4, 0, 0), & H(e_2)(x_4, y_1) = (0.5, 0.7, 0.7, 0, 0), \\
H(e_2)(x_4, y_2) = (0.4, 0.3, 0.6, 0, 0), & H(e_2)(x_4, y_3) = (0.5, 0.9, 0.4, 0, 0), \\
H(e_3)(x_1, y_1) = (0.2, 0.5, 0.6, 0, 0), & H(e_3)(x_1, y_2) = (0.2, 0.5, 0.5, 0, 0), \\
H(e_3)(x_1, y_3) = (0.3, 0.5, 0.7, 0, 0), & H(e_3)(x_2, y_1) = (0.2, 0.5, 0.6, 0, 0), \\
H(e_3)(x_2, y_2) = (0.2, 0.7, 0.5, 0, 0), & H(e_3)(x_2, y_3) = (0.5, 0.5, 0.7, 0, 0), \\
H(e_3)(x_3, y_1) = (0.2, 0.5, 0.3, 0, 0), & H(e_3)(x_3, y_2) = (0.2, 0.9, 0.3, 0, 0), \\
H(e_3)(x_3, y_3) = (0.5, 0.5, 0.3, 0, 0), & H(e_3)(x_4, y_1) = (0.2, 0.1, 0.5, 0, 0), \\
H(e_3)(x_4, y_2) = (0.2, 0.1, 0.5, 0, 0), & H(e_3)(x_4, y_3) = (0.7, 0.1, 0.5, 0, 0).
\end{array}$$

Step 3: For each $e_i \in A$: Form the i-projections $(F(e_i), X)$ and $(G(e_i), Y)$ as shown in Table 5 and Table 6.

Table 5. i-projections $(F(e_i), X)$; $i \in \{1, 2, 3\}$

	1−projection			2−projection			3−projection		
X/A	e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3
x_1	0.3	0.6	0.4	0.1	0.3	0.5	0.3	0.5	0.7
x_2	0.4	0.5	0.8	0.8	0.6	0.4	0.5	0.7	0.9
x_3	0.1	0.5	0.2	0.3	0.5	0.6	0.5	0.9	0.3
x_4	0.9	0.7	0.5	0.5	0.9	0.7	0.7	0.1	0.5

Table 6. i-projections $(G(e_i), Y)$; $i \in \{1, 2, 3\}$

	1-projection			2-projection			3-projection		
Y/A	e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3
y_1	0.2	0.6	0.5	0.8	0.7	0.9	0.2	0.5	0.6
y_2	0.9	0.5	0.3	0.4	0.3	0.6	0.2	0.9	0.5
y_3	0.6	0.5	0.1	0.7	0.9	0.4	0.9	0.5	0.7

Step 4: For each $e_i \in A$: Form the i-projections $(H(e_i), X \times Y)$ of the Γ-fuzzy soft relation (H, A) as shown in Table 7.

Table 7. i-projections $(H(e_i), X \times Y)$; $i \in \{1, 2, 3\}$

	1-projection			2-projection			3-projection		
$X \times Y$	e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3
(x_1, y_1)	0.2	0.6	0.4	0.2	0.6	0.4	0.2	0.5	0.6
(x_1, y_2)	0.3	0.5	0.3	0.1	0.3	0.5	0.2	0.5	0.5
(x_1, y_3)	0.3	0.5	0.1	0.1	0.3	0.4	0.3	0.5	0.7
(x_2, y_1)	0.2	0.5	0.5	0.8	0.6	0.4	0.2	0.5	0.6
(x_2, y_2)	0.4	0.5	0.3	0.4	0.3	0.4	0.2	0.7	0.5
(x_2, y_3)	0.4	0.5	0.1	0.7	0.6	0.4	0.5	0.5	0.7
(x_3, y_1)	0.1	0.5	0.2	0.3	0.5	0.6	0.2	0.5	0.3
(x_3, y_2)	0.1	0.5	0.2	0.3	0.3	0.6	0.2	0.9	0.3
(x_3, y_3)	0.1	0.5	0.1	0.3	0.5	0.4	0.5	0.5	0.3
(x_4, y_1)	0.2	0.6	0.5	0.5	0.7	0.7	0.2	0.1	0.5
(x_4, y_2)	0.9	0.5	0.3	0.4	0.3	0.6	0.2	0.1	0.5
(x_4, y_3)	0.6	0.5	0.1	0.5	0.9	0.4	0.7	0.1	0.5

Step 5: For each $e_i \in A$: Construct the comparison table for the i-projections $(H(e_i), X \times Y)$, as shown in Table 8, Table 9, and Table 10, where the ordered pair (x_i, y_j) is denoted by $x_i y_j$.

Table 8. Comparison table for $(H(e_1), X \times Y)$

$X \times Y$	x_1y_1	x_1y_2	x_1y_3	x_2y_1	x_2y_2	x_2y_3	x_3y_1	x_3y_2	x_3y_3	x_4y_1	x_4y_2	x_4y_3
x_1y_1	3	2	2	2	2	2	3	3	3	2	2	2
x_1y_2	1	3	3	2	2	2	3	3	3	1	2	2
x_1y_3	1	2	3	2	1	2	2	2	3	1	1	2
x_2y_1	2	2	2	3	2	2	3	3	3	2	2	2
x_2y_2	1	3	3	2	3	3	3	3	3	1	2	2
x_2y_3	1	2	3	2	2	3	2	2	3	1	1	2
x_3y_1	0	1	2	1	1	2	3	3	3	0	1	2
x_3y_2	0	1	2	1	1	2	3	3	3	0	1	2
x_3y_3	0	1	2	1	1	2	2	2	3	0	1	2
x_4y_1	3	2	2	3	2	2	3	3	3	3	2	2
x_4y_2	1	3	3	2	3	3	3	3	3	1	3	3
x_4y_3	1	2	3	2	2	3	2	2	3	1	1	3

Table 9. Comparison table for $(H(e_2), X \times Y)$

$X \times Y$	x_1y_1	x_1y_2	x_1y_3	x_2y_1	x_2y_2	x_2y_3	x_3y_1	x_3y_2	x_3y_3	x_4y_1	x_4y_2	x_4y_3
x_1y_1	3	3	3	1	2	1	0	1	1	0	1	1
x_1y_2	3	3	3	1	2	1	0	1	1	0	1	1
x_1y_3	2	2	3	1	2	1	0	1	1	0	1	1
x_2y_1	2	2	3	3	3	3	2	2	3	1	2	2
x_2y_2	2	2	3	1	3	1	1	2	2	0	2	1
x_2y_3	2	2	3	2	3	3	2	2	3	1	2	2
x_3y_1	3	3	3	1	2	1	3	3	3	0	2	1
x_3y_2	3	3	3	1	2	1	2	3	2	0	2	1
x_3y_3	2	2	3	1	2	1	2	2	3	0	1	1
x_4y_1	3	3	3	2	3	2	3	3	3	3	3	2
x_4y_2	3	3	3	1	3	1	2	3	2	0	3	1
x_4y_3	2	2	3	2	3	2	2	2	3	2	2	3

Step 6: For each $e_i \in A$: Compute the row and column-sums $W^i = \sum r^{x_l}_{pj}$ and $C^i = \sum r^{x_l}_{jp}$ of the comparison tables, as shown in Table 11, Table 12 and Table 13.

Step 7: For each $e_i \in A$: Compute the score values $S^i = W^i - C^i$.

Step 8: Find the maximum value of S^i, for all $e_i \in A$:
$\max\{S^1\} = 17$ is corresponding to the suitable managerial pair (x_4, y_1).
$\max\{S^2\} = 26$, also is corresponding to the suitable managerial pair (x_4, y_1).
$\max\{S^3\} = 19$ is corresponding to the suitable managerial pair (x_2, y_3).

Step 9: Choose the good suitable managerial Pair:
If the HR selects the team based on "`Technical Expertise`" or "`Project Management Skills`, then (x_4, y_1).
If the HR selects the team based on "`Innovation and Creativity`, then (x_2, y_3).

Table 10. Comparison table for $(H(e_3), X \times Y)$

$X \times Y$	x_1y_1	x_1y_2	x_1y_3	x_2y_1	x_2y_2	x_2y_3	x_3y_1	x_3y_2	x_3y_3	x_4y_1	x_4y_2	x_4y_3
x_1y_1	3	3	1	3	2	1	3	2	2	3	3	2
x_1y_2	2	3	1	2	2	1	3	2	2	3	3	2
x_1y_3	3	3	3	3	2	2	3	2	2	3	3	2
x_2y_1	3	3	1	3	2	1	3	2	2	3	3	2
x_2y_2	2	3	1	2	3	1	3	2	2	3	3	2
x_2y_3	3	3	3	3	2	3	3	2	3	3	3	2
x_3y_1	2	2	1	2	1	1	3	2	2	2	2	1
x_3y_2	2	2	1	2	2	1	3	3	2	2	2	1
x_3y_3	2	2	2	2	1	2	3	2	3	2	2	1
x_4y_1	1	2	0	1	2	0	2	2	1	3	3	2
x_4y_2	1	2	0	1	2	0	2	2	1	3	3	2
x_4y_3	1	2	1	2	2	1	2	2	2	3	3	3

Step 10: Output the selected good mixed suitable managerial pair according to the chosen criteria: The HR chooses the good suitable managerial pairs according to all parameters from the set $\{(x_2, y_3), (x_4, y_1)\}$.

Table 11. Comparison of Row-Sums and Column-Sums for $(H(e_1), X \times Y)$

Row-Sums	**Column-Sums**	**Score Values**
28	14	14
27	24	3
22	30	-8
28	23	5
29	22	7
24	28	-4
19	32	-13
19	32	-13
17	36	-19
30	13	17
31	19	12
25	26	-1

TABLE 12. Comparison of Row-Sums and Column-Sums for $(H(e_2), X \times Y)$

Row-Sums	**Column-Sums**	**Score Values**
17	30	-13
17	30	-13
15	36	-21
28	17	11
20	30	-10
27	18	9
25	19	6
23	25	-2
20	27	-7
33	7	26
25	22	3
28	17	11

TABLE 13. Comparison of Row-Sums and Column-Sums for $(H(e_3), X \times Y)$

Row-Sums	**Column-Sums**	**Score Values**
28	25	3
26	30	-4
31	15	16
28	26	2
27	23	4
33	14	19
21	33	-12
23	25	-2
24	24	0
19	33	-14
19	33	-14
24	22	2

4. Clarification on Γ-fuzzy soft relations vs. I-fuzzy soft relations in decision-making

While the classical I-fuzzy soft relation approach evaluates candidate pairs using scalar degrees under each parameter, the Γ-fuzzy soft relation offers a more nuanced representation. It preserves parameter-wise evaluations as vectors in $[0,1]^r$, allowing better specialization and interpretation of each parameter's effect on the decision. In addition to that in traditional fuzzy soft sets, a mapping $F : A \to I^X$ assigns to each parameter $e \in A$ a fuzzy subset $F(e) : X \to [0,1]$. This means that each object $x \in X$ receives a single membership value under each parameter e. However, in decision-making scenarios where multiple parameters jointly affect the outcome, this structure lacks the ability to represent interactions between parameters explicitly. In contrast, a Γ-fuzzy soft set employs mappings of the form $F(e) : X \to \Gamma$, where $\Gamma = [0,1]^{|E|}$. Here, the membership of each $x \in X$ is a vector representing degrees of association with each parameter in E. This richer representation allows decision-makers to analyze each parameter's influence separately and then aggregate the results systematically.

The following is as an alternative Example, which illustrates how classical I-fuzzy soft relation is used in Decision-Making.

Example 4.1. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ be sets of male and female managerial candidates, respectively. Let $A = \{e_1, e_2, e_3\}$ represent evaluation parameters:

- e_1: Experience
- e_2: Communication skills
- e_3: Leadership ability

Step 1: Define fuzzy soft sets (F, A) and (G, A) over X and Y, respectively:

$F(e_1)$	x_1	x_2	x_3
	0.6	0.8	0.4

$F(e_2)$	x_1	x_2	x_3
	0.5	0.7	0.6

$F(e_3)$	x_1	x_2	x_3
	0.7	0.6	0.5

$G(e_1)$	y_1	y_2	y_3
	0.7	0.5	0.6

$G(e_2)$	y_1	y_2	y_3
	0.6	0.8	0.4

$G(e_3)$	y_1	y_2	y_3
	0.8	0.5	0.7

Step 2: Construct the fuzzy soft relation $H : A \to I^{X \times Y}$ by:

$$H(e)(x_i, y_j) = \min(F(e)(x_i), G(e)(y_j))$$

Step 3: Form the relation tables $H(e_k),\ k = 1, 2, 3$.
For e_1:

$H(e_1)$	y_1	y_2	y_3
x_1	0.6	0.5	0.6
x_2	0.7	0.5	0.6
x_3	0.4	0.4	0.4

Similarly, compute and tabulate $H(e_2)$ and $H(e_3)$:

$H(e_2) =$

	y_1	y_2	y_3
x_1	0.5	0.5	0.4
x_2	0.6	0.7	0.4
x_3	0.6	0.6	0.4

$H(e_3) =$

	y_1	y_2	y_3
x_1	0.7	0.5	0.7
x_2	0.6	0.5	0.6
x_3	0.5	0.5	0.5

Step 4: Compute score values for each pair (x_i, y_j):

$$S(x_i, y_j) = \frac{1}{3} \sum_{k=1}^{3} H(e_k)(x_i, y_j)$$

Pair	Score
(x_1, y_1)	$\frac{0.6+0.5+0.7}{3} = 0.6$
(x_1, y_2)	$\frac{0.5+0.5+0.5}{3} = 0.5$
(x_1, y_3)	$\frac{0.6+0.4+0.7}{3} \approx 0.567$
(x_2, y_1)	$\frac{0.7+0.6+0.6}{3} = 0.633$
(x_2, y_2)	$\frac{0.5+0.7+0.5}{3} = 0.567$
(x_2, y_3)	$\frac{0.6+0.4+0.6}{3} = 0.533$
(x_3, y_1)	$\frac{0.4+0.6+0.5}{3} = 0.5$
(x_3, y_2)	$\frac{0.4+0.6+0.5}{3} = 0.5$
(x_3, y_3)	$\frac{0.4+0.4+0.5}{3} \approx 0.433$

Step 5: Choose the pair(s) with the highest score.

$$\max S(x_i, y_j) = 0.633 \quad \text{for pair } (x_2, y_1).$$

Hence, the best managerial pair according to this classical fuzzy soft relation model is (x_2, y_1).

This serves as a comparison baseline to evaluate the additional capabilities and refinements offered by the Γ-fuzzy soft relation method.

5. Conclusion

In this paper, we introduced a generalized model of fuzzy soft relations by extending the classical scalar framework $I = [0,1]$ to a parameter-wise structure $\Gamma = [0,1]^{|E|}$. This generalization enables each parameter in a decision-making environment to contribute distinctly and explicitly through vector-valued membership degrees. We developed the corresponding theoretical foundation based on a newly defined Γ-lattice and proposed a dedicated decision-making algorithm that leverages this enhanced structure.

Through comprehensive numerical examples, we demonstrated how the proposed Γ-fuzzy soft relation model preserves detailed parameter-specific influence, which is otherwise aggregated and obscured in classical fuzzy soft relations. For clear contrast, we presented a full parallel example under the classical framework using the same data. This side-by-side comparison confirmed the advantage of our model in providing greater interpretability and parametric sensitivity.

The conclusion drawn from this generalization is quite insightful and can be summarized as follows:

(1) When the parameter set E consists of only one element, the Γ-fuzzy soft relation reduces to the traditional fuzzy case with scalar values in $I = [0,1]$. In this case, the application coincides with the classical I-fuzzy decision-making method.

(2) When E includes two or more parameters, and the interval $I = [0,1]$ is still used for fuzzification, the relation behaves like a standard I-fuzzy soft relation, involving aggregated effects of multiple parameters.

(3) The true benefit of our proposed Γ-fuzzy soft relation arises when each element of E is treated independently in the construction of $\Gamma = [0,1]^{|E|}$. This

enables a finer, parameter-wise evaluation and allows for more nuanced and flexible decision-making.

This work not only proposes a refined decision-making model but also paves the way for extensions involving interval-valued, hesitant, or intuitionistic fuzzy soft sets. Thus, the proposed Γ-fuzzy soft relation framework enhances both theoretical expressiveness and practical decision-support capabilities in complex, multi-criteria environments.

References

[1] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353. http://dx.doi.org/10.1016/S0019-9958(65)90241-X

[2] D. A. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (1999) 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5

[3] P. K. Maji, A. R. Roy and R. Biswas, Fuzzy soft sets, Journal of Fuzzy Mathematics 9 (2001) 589–602. [GoogleScholar]

[4] P. Majumdar and S. K. Samanta, Generalized fuzzy soft sets, Computer Mathematics Applications 59 (2010) 1425–1432. https://doi.org/10.1016/j.camwa.2009.12.006

[5] H. Aktaş and N. Çağman, Soft sets and soft groups, Inform. Sci. 177 (13) (2007) 2726–2735. https://doi.org/10.1016/j.ins.2006.12.008

[6] Y. Zou and Z. Xiao, Data analysis approaches of soft sets under incomplete information, Knowledge-Based System 21 (2008) 941–945. DOI:10.1016/j.knosys.2008.04.004

[7] D. S. Hooda and v. Raich, Fuzzy set theory and fuzzy controller, Narora Publishing House, New Delhi, 2015. https://www.amazon.in/Fuzzy-Set-Theory-Controller/dp/1842659359

[8] H. Aktaşand N. Çağman, Soft decision-making methods based on fuzzy sets and soft sets. Journal of Intelligent and Fuzzy Systems 30 (2016) 2797–2803. [GoogleScholar]

[9] P. K. Maji, A. R. Roy and R. Biswas, An application of sets in a decision-making problem, Computers Mathematics with Applications 44 (8-9) (2002) 1077–1083. https://doi.org/10.1016/S0898-1221(02)00216-X

[10] Dusmanta Kumar Sut, An application of fuzzy soft relation in decision-making problems, International Journal of Mathematics Trends and Technology 3 (2) (2012) 50–53. https://ijmttjournal.org/archive/ijmtt-v3i2p503

[11] A. R. Roy and P. K. Maji, A Fuzzy soft set theoretic approach to decision-making problems, Journal of Computational and Applied Mathematics 203 (2) (2007) 412–418. https://doi.org/10.1016/j.cam.2006.04.008

[12] B. Ahmad B, A. Kharal, On Fuzzy Soft Sets. Advances in Fuzzy Systems (2009) 1-6. https://doi.org/10.1155/2009/586507

[13] D. Chen, E. C. C. Tsang, D. S. Yeung and X. Wang, The parameterization reduction of soft sets and its applications, Computers and Mathematics with Applications 49 (2005) 757–763. DOI:10.1016/j.camwa.2004.10.036

[14] P. K. Maji and A. R. Roy, Soft set theory, Computers and Mathematics with Applications 45 (2003) 555–562. http://dx.doi.org/10.1016/S0898-1221(03)00016-6

[15] D. Pei and D. Miao, From soft sets to information systems. In: IEEE International Conference on Granular Computing. Beijing, China 2 (2005) 617–621. Available at: https://doi.org/10.1109/GRC.2005.1547365

[16] J. Močkoř and P. Hurtík, Approximations of fuzzy soft sets by fuzzy soft relations with image processing application, Soft Comput. 25 (2021) 6915–6925. DOI:10.1007/s00500-021-05769-3

[17] Garrett Birkhoff, Lattice Theory, American Mathematical Soc. 25 (2) (1940).

[18] J. Močkoř and P. Hurtik, Approximations of fuzzy soft sets by fuzzy soft relations with image processing application, Soft Computing, 25 (2021) 6915–6925. https://doi.org/10.1007/s00500-021-05769-3

[19] N. M. Kandil, M. A. Hassan and M. A. Ali, Hesitant fuzzy soft multisets and their applications in decision-making problems, Soft Computing 24 (6) (2020) 4223–4232. https://doi.org/10.1007/s00500-019-04187-w

[20] P. Jayaraman, D. Divya and K. Vinothkumar, Application of picture fuzzy soft relations in multi-attribute decision making, Journal of University of Shanghai for Science and Technology Check Vol.number. Issuue Number and pages (2023). https://jusst.org/wp-content/uploads/2023/01/Application-of-Picture-Fuzzy-Soft-Relations.pdf

[21] A. Mujtaba, M. Shoaib, A. Abbas and S. Ali, Q-Neutrosophic soft relation and its application in decision making, *Entropy* 24 (12) (2022) 1755. https://doi.org/10.3390/e24121755

[22] I. Djurović, M. Petrović, and M. Savić, Decision-making algorithm based on the energy of interval-valued fuzzy soft sets, arXiv preprint arXiv:2405.15801 (2024). https://arxiv.org/abs/2405.15801

[23] K. A. Blowi, S. Abdullah, H. Al-Muhtaseb and N. Kharma, Decision making based on fuzzy soft sets and its application in COVID-19, Intelligent Automation & Soft Computing 30 (3) (2021) 961–972. https://doi.org/10.32604/iasc.2021.017193

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